

SINGULAR LOCI OF COMINUSCULE SCHUBERT VARIETIES

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ABSTRACT. Let $X = G/P$ be a cominuscule rational homogeneous variety. Equivalently, X admits the structure of a compact Hermitian symmetric space. I give a uniform description (that is, independent of type) of the irreducible components of the singular locus of a Schubert variety $Y \subset X$ in terms of representation theoretic data. The result is based on a recent characterization of the Schubert varieties by an integer $\mathbf{a} \geq 0$ and a marked Dynkin diagram. Corollaries include: (1) the variety is smooth if and only if $\mathbf{a} = 0$; (2) if G of Type ADE, then the singular locus occurs in codimension at least three.

1. INTRODUCTION

Let $X = G/P$ be a cominuscule rational homogeneous variety. (Equivalently, X admits the structure of a compact Hermitian symmetric space. An example is the Grassmannian $\mathrm{Gr}(k, n)$ of k -planes in complex n -space.) The main result (Theorem 3.3) of this paper is a uniform (independent of G) description of the irreducible components of the singular loci of Schubert varieties $Y \subset X$.

Context and related results. Type-specific descriptions in the case that X is classical (G is of type ABCD) have been known for some time: the type A case (X is a Grassmannian) since the 1970s, and the rest by 1990; see [1, Section 9.3] and the references therein. Those descriptions are given in terms of partitions. In contrast, the type-free characterization of Theorem 3.3 is given by representation theoretic data.

The cominuscule X are closely related to the minuscule rational homogeneous varieties. Indeed, with the exception of the quadric hypersurface $Q^{2n-1} = B_n/P_1$ and the Lagrangian Grassmannian $\mathrm{LG}(n, 2n) = C_n/P_n$, every irreducible cominuscule X is minuscule. Theorem 3.3 compliments descriptions of the singular loci of Schubert varieties in minuscule \mathcal{X} by M. Brion and P. Polo [3], and by N. Perrin [12]. (Brion and Polo also study singularities of cominuscule Schubert varieties, but – so far as I can discern – stop short of a complete description of the irreducible components.)

S. Kumar gave a type-free smoothness criterion for Schubert varieties in \mathcal{G}/\mathcal{B} , where \mathcal{B} is a Borel subgroup of a complex, simply connected, semisimple group \mathcal{G} [8]. More generally, it is an important open question to explicitly describe the singular loci of Schubert varieties in an arbitrary rational homogeneous variety \mathcal{G}/\mathcal{P} , see [1]. In the case of the full flag variety $\mathrm{SL}(V)/\mathcal{B}$, this has been done (independently) by several people [2, 5, 6, 11, 10].

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Contents. In Section 2 we review Schubert varieties, and their representation theoretic characterization by an integer $\mathbf{a} \geq 0$ and a marking \mathbf{J} of the Dynkin diagram of G . The main theorem and subsequent corollaries are discussed in Section 3. Corollaries 3.5 and 3.6 describe the relationship between the integer $\mathbf{a} = \mathbf{a}(Y)$ and the number of irreducible components in $\text{Sing}(Y)$; for example, Y is smooth if and only if $\mathbf{a} = 0$; if $\mathbf{a} = 1$, then $\text{Sing}(Y)$ is irreducible. Corollary 3.7 gives lower bounds on the codimension of the singular locus, and characterizes those Schubert varieties for which the bound is realized. The main result is proved in Section 4.

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1.1. Use of LiE. There are two exceptional, irreducible compact Hermitian symmetric spaces: the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7 . The software [9] is used to perform computations and verify some results for these two cominuscule varieties. (Sections 3.1 and A.4.) The code I wrote applies to any cominuscule variety; that is, it is *not* specialized to the exceptional cases. This allowed me to test the code by applying it to classical cominuscule varieties (of low rank – the code requires that the rank of G be specified). The outputs are consistent with the “by hand” results obtained for the classical cases. This provides some confidence for the accuracy of the code and the corresponding claims in the two exceptional cases.

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2. REVIEW

2.1. Notation and background. The present article is founded on a result of [14]. With the exception noted in Remark 2.9, I will use the notation of that paper. To streamline the presentation, I will give a laconic review of the discussion of rational homogeneous varieties, their Schubert subvarieties, grading elements and Hasse diagrams in [14, Sections 2.1-2.4 and 3.1]. Briefly, G is a complex simple Lie group. A choice of Cartan and Borel subgroups $H \subset B$ has been fixed, $P \supset B$ is a maximal parabolic subgroup associated with a cominuscule root, and $X = G/P$ is the corresponding cominuscule variety. The associated

Lie algebras are denoted $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$. Let W denote the Weyl group of \mathfrak{g} , and $W_{\mathfrak{p}}$ the Weyl group of the reductive component in the Levi decomposition of \mathfrak{p} . The Hasse diagram $W^{\mathfrak{p}}$ is the set of minimal length representatives of the left-coset space $W_{\mathfrak{p}} \backslash W$, and indexes the Schubert classes. Let

$$o = P/P \in X = G/P.$$

Given $w \in W^{\mathfrak{p}}$, the Zariski closure

$$Y_w := \overline{Bw^{-1} \cdot o}$$

is a Schubert variety. Any G -translate of the Schubert variety Y_w will be referred to as a *Schubert variety of type w* . Let $\xi_w = [Y_w] \in H_{2|w|}(X, \mathbb{Z})$ denote the corresponding Schubert class.

Let $\{Z_1, \dots, Z_r\}$ be the basis of \mathfrak{h} dual to the simple roots $\Sigma = \{\alpha_1, \dots, \alpha_r\}$. Let α_i be the simple root associated with the cominuscule \mathfrak{p} . Then

$$(2.1) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \quad \text{where} \quad \mathfrak{g}_k := \{A \in \mathfrak{g} \mid [Z_i, A] = kA\},$$

is the Z_i -graded decomposition of the Lie algebra \mathfrak{g} . Moreover,

$$(2.2) \quad \mathfrak{p} = \mathfrak{g}_1 \oplus \mathfrak{g}_0,$$

and \mathfrak{g}_0 is the reductive component of the parabolic subalgebra \mathfrak{p} .

Remark 2.3. As a graded decomposition, we have $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$. In particular, the subspaces $\mathfrak{g}_{\pm 1}$ are both \mathfrak{g}_0 -modules, and abelian subalgebras of \mathfrak{g} .

Equation (2.2) implies

$$T_o X \simeq \mathfrak{g}_{-1}$$

as an \mathfrak{g}_0 -module.

Notation. Let Δ denote the set of roots of \mathfrak{g} . Given $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denote the corresponding root space. Given any subset $\mathfrak{s} \subset \mathfrak{g}$, let

$$\Delta(\mathfrak{s}) = \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{s}\}.$$

Given a subset U of a vector space, let $\langle U \rangle$ denote the linear span.

2.2. The characterization of Schubert varieties. This section is a concise review of the characterization of Schubert classes ξ_w by an integer $\mathbf{a}(w) \geq 0$ and a marking $\mathbf{J}(w)$ of the Dynkin diagram. (The marking is equivalent to a choice of simple roots from $\Sigma \setminus \{\mathbf{i}\}$.) For more detail see [14]. Given $w \in W^{\mathfrak{p}}$, define

$$(2.4) \quad \Delta(w) = w\Delta^- \cap \Delta^+ \subset \Delta(\mathfrak{g}_1) \quad \text{and} \quad \mathfrak{n}_w = \bigoplus_{\alpha \in \Delta(w)} \mathfrak{g}_{-\alpha} \subset \mathfrak{g}_{-1}.$$

Let $N_w = \exp(\mathfrak{n}_w)$. Then

$$(2.5) \quad X_w := \overline{N_w \cdot o} = wY_w$$

is a Schubert variety of type w .

By work of Kostant [7], the ℓ -th exterior power $\bigwedge^\ell \mathfrak{g}_{-1}$ decomposes into irreducible \mathfrak{g}_0 -modules \mathbf{I}_w , which are indexed by elements of $W^{\mathbf{p}}$ of length ℓ

$$\bigwedge^\ell \mathfrak{g}_{-1} = \bigoplus_{\substack{w \in W^{\mathbf{p}} \\ |w| = \ell}} \mathbf{I}_w.$$

The highest weight line in $\mathbb{P}\mathbf{I}_w$ is \mathbf{n}_w . Let $1 \in W^{\mathbf{p}}$ be the identity, and let $w_0 \in W^{\mathbf{p}}$ be the longest element. Then $X_1 = o$ and $X_{w_0} = X$, and \mathbf{I}_1 and \mathbf{I}_{w_0} are trivial \mathfrak{g}_0 -modules. Assume $w \in W^{\mathbf{p}} \setminus \{1, w_0\}$. Let $\mathfrak{q}_w \subset \mathfrak{g}_0$ be the stabilizer of the highest weight line \mathbf{n}_w . Then there is a subset $J(w) \subset \{1, \dots, r\} \setminus \{i\}$ with the property that the Lie algebra \mathfrak{q}_w is given by $\mathfrak{q}_w = \mathfrak{g}_{0, \geq 0}$, where

$$(2.6) \quad \mathfrak{g}_{k, \ell} := \{A \in \mathfrak{g}_k \mid [Z_w, A] = \ell A\} \quad \text{and} \quad Z_w := \sum_{j \in J(w)} Z_j.$$

We call $\mathfrak{g} = \bigoplus \mathfrak{g}_{k, \ell}$ the (Z_i, Z_w) -bigraded decomposition of \mathfrak{g} . It is a simple consequence of standard representation theory that $k\ell < 0$ forces $\mathfrak{g}_{k, \ell} = 0$. The following is [14, Proposition 3.9].

Proposition 2.7 ([14]). *Let $w \in W^{\mathbf{p}} \setminus \{1, w_0\}$. There exists an integer $\mathbf{a} = \mathbf{a}(w) \geq 0$ such that $\Delta(w) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid \alpha(Z_w) \leq \mathbf{a}\}$. Equivalently,*

$$(2.8) \quad \mathbf{n}_w = \mathfrak{g}_{-1, 0} \oplus \cdots \oplus \mathfrak{g}_{-1, -\mathbf{a}}.$$

Remarks. \circ Since $\xi_w = [X_w]$, and X_w is determined by $\Delta(w)$, the pair $\mathbf{a}(w), J(w)$ characterizes ξ_w , when $w \in W^{\mathbf{p}} \setminus \{1, w_0\}$.

- \circ In the case that X is a Grassmannian (Type A), the Schubert varieties are indexed by partitions. Proposition 2.7 describes the relationship between the (\mathbf{a}, J) and partition descriptions.
- \circ By [14, Proposition 3.19], the Schubert variety X_w is smooth if and only if $\mathbf{a}(w) = 0$. In particular, the proposition generalizes the description of smooth Schubert varieties by sub-diagrams. For more on the relationship between the integer $\mathbf{a}(w)$ and $\text{Sing}(X_w)$, see Corollary 3.6.
- \circ A tableau-esque analog of Proposition 2.7 is given by H. Thomas and A. Yong in [15, Proposition 2.1].

A complete list of the $\mathbf{a}(w), J(w)$ that occur is given by [14, Corollary 3.17]. In the appendix we review the geometric description of the Schubert variety X_w associated with a pair $\mathbf{a}(w), J(w)$, in the case that the cominuscule G/P is classical and irreducible. The values $\mathbf{a}(w), J(w)$ corresponding to Schubert varieties in the two exceptional cominuscule varieties are given by Figures 1 and 2.

Remark 2.9. I follow the notation of [14], with the following exception. In [14], we uniformly write $J = \{j_1 < \cdots < j_p\}$. Here, it is convenient to reorder the j_ℓ in some cases.

3. SINGULAR LOCUS

For this section we fix, once and for all, $w \in W^{\mathbf{p}} \setminus \{1, w_0\}$. The singular locus $\text{Sing}(Y_w)$ is a union of Schubert subvarieties $Y_{w'} \subset Y_w$. Let $\text{Sing}_w \subset W^{\mathbf{p}}$ be the subset indexing the

irreducible components of $\text{Sing}(Y_w)$, so that

$$\text{Sing}(Y_w) = \bigcup_{w' \in \text{Sing}_w} Y_{w'}.$$

Definition. Let $\Pi_{1,a-1} := \{\varepsilon \in \Delta(\mathfrak{g}_{1,a-1}) \mid \varepsilon + \alpha \notin \Delta \ \forall \ \alpha \in \Delta^+(\mathfrak{g}_{0,0})\}$. Equivalently, $\Pi_{1,a-1}$ is the set of highest weights associated with the $\mathfrak{g}_{0,0}$ -module $\mathfrak{g}_{1,a-1}$.

Let $\varepsilon \in \Pi_{1,a-1}$. Define

$$(3.1) \quad \begin{aligned} \Delta(w, \varepsilon) &:= \{\varepsilon\} \sqcup \{\nu \in \Delta(\mathfrak{g}_{1,a}) \mid \nu - \varepsilon \in \Delta(\mathfrak{g}_{0,1})\} \\ &= \{\varepsilon\} \sqcup \{\Delta \cap (\varepsilon + \Delta(\mathfrak{g}_{0,1}))\}. \end{aligned}$$

Lemma 3.2. *There exists $w_\varepsilon \in W^\mathfrak{p}$ such that $\Delta(w_\varepsilon) = \Delta(w) \setminus \Delta(w, \varepsilon)$.*

Theorem 3.3. *The roots $\Pi_{1,a-1}$ are in bijective correspondence with the irreducible components of the singular locus $\text{Sing}(Y_w)$. Explicitly,*

$$\text{Sing}_w = \{w_\varepsilon \mid \varepsilon \in \Pi_{1,a-1}\}.$$

The lemma and theorem are proved in Section 4.2.

Example. Theorem 3.3 generalizes the well-known descriptions [1, Section 9.3] of the singular locus of Y_w in the case that G is classical. Here is a simple illustration in the case that $X = \text{Gr}(5, 11)$ is a Grassmannian. Assume the notations and definitions of Section A.1. In particular, the subalgebra $\mathfrak{g}_{-1} \subset \mathfrak{sl}_n \mathbb{C}$ is spanned by $\{e_k^j \mid 1 \leq j \leq 5, 6 \leq k \leq 11\}$. The basis element $e_{k+1}^j \in \mathfrak{g}_{-1}$ is a root vector for the root

$$-\alpha_{jk} := -(\alpha_j + \cdots + \alpha_k).$$

Fix $w \in W^\mathfrak{p}$ with $\mathbf{a}(w) = 2$ and $\mathbf{J}(w) = \{2, 3, 6, 8, 10\}$. The corresponding partition is $\pi = (5^2, 3, 1^2)$, see Proposition A.1. Any element $s_j^k e_k^j$ of \mathfrak{g}_{-1} may be represented by a matrix (s_k^j) where $1 \leq j \leq 5$ and $6 \leq k \leq 11$. The corresponding Z_w -degrees are given by

$$- \left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 1 & 1 \\ 3 & 3 & 2 & 1 & 1 \\ \hline 4 & 4 & 3 & 2 & 2 \\ 4 & 4 & 3 & 2 & 2 \\ \hline 5 & 5 & 4 & 3 & 3 \end{array} \right].$$

So the subspace $\mathfrak{n}_w \subset \mathfrak{g}_{-1}$ is represented by

$$\mathfrak{n}_w = \left(\begin{array}{cc|cc} s_1^6 & s_2^6 & s_3^6 & s_4^6 & s_5^6 \\ 0 & 0 & s_3^7 & s_4^7 & s_5^7 \\ 0 & 0 & s_3^8 & s_4^8 & s_5^8 \\ \hline 0 & 0 & 0 & s_4^9 & s_5^9 \\ 0 & 0 & 0 & s_4^{10} & s_5^{10} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Define a filtration $F^3 \subset F^6 \subset F^{10} \subset \mathbb{C}^{11}$ by $F^3 = \langle e_1, e_2, e_6 \rangle$, $F^6 = \langle F^3, e_3, e_7, e_8 \rangle$ and $F^{10} = \langle F^6, e_4, e_5, e_9, e_{10} \rangle$. Then (2.5) yields $Y_w = w^{-1}X_w$, with

$$X_w = \{E \in \text{Gr}(5, 11) \mid \dim(E \cap F^3) \geq 2, \dim(E \cap F^6) \geq 3, \dim(E \cap F^{10}) \geq 5\}.$$

The subalgebra $\mathfrak{g}_{0,0}$ is identified with the diagonal block matrices $\text{diag}(2, 1, 2, 1, 2, 2, 1)$ in \mathfrak{sl}_{11} . The $\mathfrak{g}_{0,0}$ -module $\mathfrak{g}_{-1,-1}$ consists of two irreducible submodules. The first is spanned by e_6^3 , with highest weight $\varepsilon_1 = -\alpha_{35}$. The second is four-dimensional with highest weight vector e_8^4 and highest weight $\varepsilon_2 = -\alpha_{47}$. We have $\Delta(w, \varepsilon_1) = \{\alpha_{15}, \alpha_{25}, \alpha_{35}, \alpha_{36}, \alpha_{37}\}$ and $\Delta(w, \varepsilon_2) = \{\alpha_{37}, \alpha_{47}, \alpha_{48}, \alpha_{49}\}$.^{*} The two irreducible components $Y_{w_1} = w_1^{-1}X_{w_1}$ and $Y_{w_2} = w_2^{-1}X_{w_2}$ of $\text{Sing}(Y_w)$ correspond to

$$\mathbf{n}_{w_1} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & 0 & s_4^6 & s_5^6 \\ \hline 0 & 0 & 0 & s_4^7 & s_5^7 \\ 0 & 0 & 0 & s_4^8 & s_5^8 \\ \hline 0 & 0 & 0 & s_4^9 & s_5^9 \\ 0 & 0 & 0 & s_4^{10} & s_5^{10} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad \text{and} \quad \mathbf{n}_{w_2} = \left(\begin{array}{c|c|c|c|c} s_1^6 & s_2^6 & s_3^6 & s_4^6 & s_5^6 \\ \hline 0 & 0 & s_3^7 & s_4^7 & s_5^7 \\ 0 & 0 & 0 & s_4^8 & s_5^8 \\ \hline 0 & 0 & 0 & 0 & s_5^9 \\ 0 & 0 & 0 & 0 & s_5^{10} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The corresponding \mathbf{a} and \mathbf{J} values are $\mathbf{a}(w_1) = 0$ and $\mathbf{J}(w_2) = \{3, 10\}$; and $\mathbf{a}(w_2) = 2$ and $\mathbf{J}_{w_2} = \{2, 4, 6, 7, 10\}$.

Corollary 3.4. *The Schubert variety Y_w is smooth if and only if $\mathbf{a}(w) = 0$. If $\mathbf{a}(w) = 1$, then $\text{Sing}(Y_w)$ is a single irreducible Schubert variety.*

Remark. The first part of the corollary was proved in [14, Proposition 3.9].

Proof. The first statement is an immediate consequence of $\mathfrak{g}_{1,\mathbf{a}-1} = \mathfrak{g}_{1,-1} = \{0\}$. The second statement is a consequence of the fact that $\mathfrak{g}_{1,\mathbf{a}-1} = \mathfrak{g}_{1,0}$ is an irreducible $\mathfrak{g}_{0,0}$ -module, see [14, Section 3.2]. \square

The following is an immediate consequence of Theorem 3.3.

Corollary 3.5. *The number of irreducible components in $\text{Sing}(Y_w)$ is the number $|\Pi_{1,\mathbf{a}-1}|$ of components in a decomposition of $\mathfrak{g}_{1,\mathbf{a}-1}$ into irreducible $\mathfrak{g}_{0,0}$ -submodules.*

From Corollary 3.5, and the (\mathbf{a}, \mathbf{J}) -characterizations of Sections A.1–A.3 we deduce

Corollary 3.6. *Let Y_w be a Schubert variety in a classical, irreducible cominusculum $X = G/P$, and let $|\text{Sing}_w|$ be the number of irreducible components in $\text{Sing}(Y_w)$.*

- (a) *If $X = \text{Gr}(\mathbf{i}, n+1)$, then $|\text{Sing}_w| = \mathbf{a}(w)$.*
- (b) *If $X = \text{LG}(n, 2n)$, then $|\text{Sing}_w| = \lceil \mathbf{a}(w)/2 \rceil$.*
- (c) *Suppose $X = \mathcal{S}_n$ and assume $\mathbf{a}(w) > 1$. Set $\mathbf{r} = \lceil \frac{1}{2}(\mathbf{a} + \alpha_{n-1}(Z_w)) \rceil$. If $1 = \mathbf{j}_{r-1} - \mathbf{j}_r$, then $|\text{Sing}_w| = \lfloor \frac{1}{2}(\mathbf{a} + \alpha_{n-1}(Z_w)) \rfloor \in \{\mathbf{r} - 1, \mathbf{r}\}$; otherwise $|\text{Sing}_w| = \mathbf{r}$.*

Remark. The singular loci of Schubert varieties in quadric hypersurfaces B_n/P_1 and D_n/P_1 are so simple that I omitted them from the corollary. See [1, Section 9.3].

^{*}Note that the associated subspaces of \mathbf{n}_w are ‘hooks’ that are added to π to obtain the partitions associated with the irreducible components of $\text{Sing}(Y_\pi)$.

Proof of Corollary 3.6(a). Adopt the notation of Appendix A.1, and assume $\mathbf{a} > 0$. Then the highest $\mathfrak{g}_{0,0}$ -weights of $\mathfrak{g}_{1,\mathbf{a}-1}$ are

$$\varepsilon = \alpha_{j_{\ell}+1} + \cdots + \alpha_{\mathbf{i}} + \cdots + \alpha_{\mathbf{k}_m-1},$$

with $0 < \ell, m$ and $\ell + m = \mathbf{a} + 1$. Thus $|\Pi_{1,\mathbf{a}-1}| = \mathbf{a}(w)$. \square

Proof of Corollary 3.6(b). Adopt the notation of Appendix A.1, and assume $\mathbf{a} > 0$. Suppose that $\mathbf{a} = 2\mathbf{s}$. Then the highest $\mathfrak{g}_{0,0}$ -weights of $\mathfrak{g}_{1,\mathbf{a}-1}$ are

$$(*) \quad \varepsilon = \alpha_{j_m+1} + \cdots + \alpha_{j_{\ell}} + 2(\alpha_{j_{\ell}+1} + \cdots + \alpha_{n-1}) + \alpha_n,$$

with $1 \leq \ell < m$ and $\ell + m = \mathbf{a} + 1 = 2\mathbf{s} + 1$. Therefore, $|\Pi_{1,\mathbf{a}-1}| = \mathbf{s} = \lceil \mathbf{a}(w)/2 \rceil$.

If $\mathbf{a} = 2\mathbf{s} - 1$, then the highest $\mathfrak{g}_{0,0}$ -weights of $\mathfrak{g}_{1,\mathbf{a}-1}$ are $(*)$ and

$$\varepsilon = 2(\alpha_{j_s+1} + \cdots + \alpha_{n-1}) + \alpha_n.$$

Thus, $|\Pi_{1,\mathbf{a}-1}| = \mathbf{s} = \lceil \mathbf{a}(w)/2 \rceil$. \square

The proof of Corollary 3.7(c), which is very similar to, though more tedious than, that of Corollary 3.7(b), is left to the reader.

3.1. The exceptional cases. The irreducible components of $\text{Sing}(Y_w)$ have been determined in the case that G is classical and P is (co)minuscule; see [1, Section 9.3]. A fourth corollary of Theorem 3.3 is an explicit description of the singular locus of the Schubert varieties in the exceptional Cayley plane E_6/P_6 and Freudenthal variety E_7/P_7 (both of which are minuscule and cominuscule). See Tables 1 and 2 on pages 8 and 9, respectively. The tables are obtained with the assistance of [9], see Section 1.1.

Key to Tables 1 and 2. Each row represents a proper ($\neq o, X$) Schubert variety Y_w of X , indexed by $w \in W^{\mathbf{p}} \setminus \{1, w_0\}$. The first column is the dimension of Y_w ; the second column expresses w as a reduced product of simple reflections, acting on the left; the third column gives the corresponding $\mathbf{a}(w) : \mathbf{J}(w)$ values, see Section 2.2; and the fourth column lists the irreducible components of the singular locus in terms of their $\mathbf{a} : \mathbf{J}$ characterization. (See also Figures 1 and 2 on pages 18 and 19, respectively.)

Remarks. From the tables, we see that:

- (a) The the irreducible components $\{Y_{w_{\varepsilon}} \mid \varepsilon \in \Pi_{1,\mathbf{a}-1}\}$ of the singular locus of a Schubert variety Y_w in E_6/P_6 or E_7/P_7 satisfy $\text{codim}_{Y_w} Y_{w_{\varepsilon}} \geq 3$.
- (b) The singular locus of a Schubert variety in the Cayley plane consists of at most one irreducible component.
- (c) The singular locus of a Schubert variety in the Freudenthal variety consists of at most two irreducible components.

TABLE 1. Schubert varieties and their singular loci in the Cayley plane.

dim	w	$\mathbf{a} : \mathbf{J}$	Sing_w
1	6	0:5	
2	65	0:4	
3	654	0:23	
4	6542	0:3	
4	6543	0:12	
5	65432	1:123	0:4
5	65431	0:2	
6	654321	1:23	0:4
6	654324	1:14	0:5
7	6543241	2:124	0:3
7	6543245	1:15	o
8	65432413	1:4	0:5
8	65432451	2:125	0:3

dim	w	$\mathbf{a} : \mathbf{J}$	Sing_w
8	65432456	0:1	
9	654324513	3:145	1:23
9	654324561	1:12	0:3
10	6543245134	2:35	0:2
10	6543245613	2:14	1:23
11	65432451342	1:5	o
11	65432456134	3:135	1:4
12	654324561342	2:15	1:4
12	654324561345	1:3	0:2
13	6543245613452	3:35	2:14
14	65432456134524	2:4	1:12
15	654324561345243	1:2	0:1

3.2. Codimension of the singular locus. Let $\varepsilon \in \Pi_{1,\mathbf{a}-1}$. By Lemma 3.2 and Theorem 3.3, the irreducible component $Y_{w_\varepsilon} \subset \text{Sing}(Y_w)$ has codimension $|\Delta(w, \varepsilon)|$. In this section we characterize the Schubert varieties Y_w for which the codimension is minimal. Zelevinskii [16] showed that every Schubert variety in the Grassmannian admits a small resolution. So, in particular, we know that $\text{codim}_{Y_w} Y_{w_\varepsilon} \geq 3$ for all $Y_w \subset \text{Gr}(\mathbf{i}, n+1)$. This inequality also holds (and is sharp) for Schubert varieties in the spinor variety $\mathcal{S}_n = D_n/P_n$. On the other hand, the Lagrangian Grassmannian admits Schubert varieties with $\text{codim}_{Y_w} Y_{w_\varepsilon} = 2$.

Corollary 3.7. *Let $X = G/P$ be cominuscule. Fix $w \in W^{\mathbf{p}} \setminus \{1, w_0\}$ with associated \mathbf{a}, \mathbf{J} . Let $\varepsilon \in \Pi_{1,\mathbf{a}-1}$.*

- (a) *Suppose $X = \text{Gr}(\mathbf{i}, n+1) = A_n/P_{\mathbf{i}}$. Then $\text{codim}_{Y_w} Y_{w_\varepsilon} \geq 3$. Assume the notation of Section A.1. Equality holds if and only if there exist $0 < \ell \leq \mathbf{p}$ and $0 < m \leq \mathbf{q}$ such that $\ell + m = \mathbf{a} + 1$ and $1 = \mathbf{j}_\ell - \mathbf{j}_{\ell+1} = \mathbf{k}_{m+1} - \mathbf{k}_m$.*
- (b) *Suppose $X = \text{LG}(n, 2n) = C_n/P_n$. Then $\text{codim}_{Y_w} Y_{w_\varepsilon} \geq 2$. Assume the notation of Section A.2. Equality holds if and only if $\mathbf{a} = 2\ell - 1 > 0$ and $1 = \mathbf{j}_\ell - \mathbf{j}_{\ell+1}$. In particular, these Y_w admit no small resolution.*
- (c) *Suppose $X = \mathcal{S}_n = D_n/P_n$. Then $\text{codim}_{Y_w} Y_{w_\varepsilon} \geq 3$. Assume the notation of Section A.3. If $\mathbf{j}_1 = n - 1$ and $\mathbf{a} = 1$, then equality holds if and only if $\mathbf{j}_2 = n - 3$; if $\mathbf{j}_1 = n - 1$ and $\mathbf{a} = 2$, then equality holds if and only if $\mathbf{j}_2 = n - 2$ and $\mathbf{j}_3 = n - 4$. In all other cases, equality holds if and only if there exist $\alpha_{n-1}(Z_w) < \ell < m$ such that $\ell + m = \mathbf{a} + 1 + \alpha_{n-1}(Z_w)$, and $1 + \delta_{\ell r} = \mathbf{j}_\ell - \mathbf{j}_{\ell+1}$ and $1 = \mathbf{j}_m - \mathbf{j}_{m+1}$.*

Recall from Section A.3 that $\mathbf{a}_{n-1}(Z_w) = 0$ if $n-1 \notin \mathbf{J}$; otherwise $\mathbf{j}_1 = n_1$ and $\mathbf{a}_{n-1}(Z_w) = 1$. Also $\mathbf{r} = \lceil \frac{1}{2}(\mathbf{a} + \alpha_{n-1}(Z_w)) \rceil$.

TABLE 2. Schubert varieties and their singular loci in the Freudenthal variety.

dim	w	$\mathbf{a} : \mathbf{J}$	Sing_w
1	7	0:6	
2	76	0:5	
3	765	0:4	
4	7654	0:23	
5	76542	0:3	
5	76543	0:12	
6	765432	1:123	0:4
6	765431	0:2	
7	7654321	1:23	0:4
7	7654324	1:14	0:5
8	76543241	2:124	0:3
8	76543245	1:15	0:6
9	765432413	1:4	0:5
9	765432451	2:125	0:3
9	765432456	1:16	o
10	7654324513	3:145	1:23
10	7654324561	2:126	0:3
10	7654324567	0:1	
11	76543245134	2:35	0:2
11	76543245613	3:146	1:23
11	76543245671	1:12	0:3
12	765432451342	1:5	0:6

dim	w	$\mathbf{a} : \mathbf{J}$	Sing_w
12	765432456134	4:1356	1:4
12	765432456713	2:14	1:23
13	7654324561342	3:156	1:4
13	7654324561345	2:36	0:2
13	7654324567134	3:135	1:4
14	76543245613452	4:356	3:146
14	76543245671342	2:15	1:4
14	76543245671345	3:136	2:35
15	765432456134524	3:46	2:126
15	765432456713452	5:1356	1:5, 2:14
15	765432456713456	1:3	0:2
16	7654324561345243	2:26	1:16
16	7654324567134524	4:146	1:5, 1:12
16	7654324567134562	3:35	2:14
17	76543245613452431	1:6	o
17	76543245671345243	3:126	1:5, 0:1
17	76543245671345624	5:346	2:15
18	765432456713452431	2:16	1:5
18	765432456713456243	4:236	2:15
18	765432456713456245	2:4	1:12
19	7654324567134562431	3:36	2:15
19	7654324567134562453	5:246	3:35

dim	w	$\mathbf{a} : \mathbf{J}$	Sing_w
20	76543245671345624531	4:46	3:35
20	76543245671345624534	3:25	1:3
21	765432456713456245341	5:256	2:4
21	765432456713456245342	1:2	0:1
22	7654324567134562453421	3:26	2:4
22	7654324567134562453413	2:5	1:3
23	76543245671345624534132	4:25	4:46
24	765432456713456245341324	3:4	3:36
25	7654324567134562453413245	2:3	2:16
26	76543245671345624534132456	1:1	1:6

Proof of Corollary 3.7(a). The elements of $\Pi_{1,\mathbf{a}-1}$ are of the form

$$\varepsilon = \alpha_{j_\ell+1} + \cdots + \alpha_{k_m-1}, \quad \text{with} \quad \ell + m = \mathbf{a} + 1 \quad \text{and} \quad 0 \leq \ell, m.$$

Both $\nu_1 = \alpha_{j_\ell}$ and $\nu_2 = \alpha_{k_m}$ are elements of $\Delta(\mathfrak{g}_{0,1})$, and ε , $\varepsilon + \nu_1$ and $\varepsilon + \nu_2$ are distinct elements of $\Delta(w, \varepsilon)$. Thus $\text{codim}_{Y_w} Y_{w_\varepsilon} = |\Delta(w, \varepsilon)| \geq 3$. Additionally, $|\Delta(w, \varepsilon)| = 3$ if and only if $1 = j_\ell - j_{\ell+1} = k_{m+1} - k_m$. \square

Proof of Corollary 3.7(b). The elements of $\Pi_{1,a-1}$ are of the form $\varepsilon' = \alpha_{j_{a+1}} + \cdots + \alpha_n$, or

$$\varepsilon = \alpha_{j_{\ell+1}} + \cdots + \alpha_{j_m} + 2(\alpha_{j_{m+1}} + \cdots + \alpha_{n-1}) + \alpha_n,$$

with $\ell + m = a + 1$ and $0 < m \leq \ell$. For ε' , observe that $\nu_1 = \alpha_{j_a} \in \Delta(\mathfrak{g}_{0,1})$ and ε' , $\varepsilon' + \nu_1 \in \Delta(w, \varepsilon')$. So $\text{codim}_{Y_w} Y_{w_{\varepsilon'}} = |\Delta(w, \varepsilon')| \geq 2$. Equality holds if and only if $1 = j_a - j_{a+1}$ and $a = 1$. (If $a > 1$, then $\nu_2 = \alpha_{j_1} + \cdots + \alpha_{n-1} \in \Delta(\mathfrak{g}_{0,1})$ and $\varepsilon' + \nu_2 \in \Delta(w, \varepsilon)$ is distinct from ε' and $\varepsilon' + \nu_1$.)

For ε with $\ell < m$, observe that both $\nu_1 = \alpha_{j_\ell}$ and $\nu_2 = \alpha_{j_m}$ are elements of $\Delta(\mathfrak{g}_{0,1})$, and ε , $\varepsilon + \nu_1$ and $\varepsilon + \nu_2$ are distinct elements of $\Delta(w, \varepsilon)$. Thus $\text{codim}_{Y_w} Y_{w_\varepsilon} = |\Delta(w, \varepsilon)| \geq 3$. If $\ell = m$, then $\nu_1 = \nu_2$. So, $\text{codim}_{Y_w} Y_{w_\varepsilon} = |\Delta(w, \varepsilon)| \geq 2$. Equality holds if and only if $1 = j_\ell - j_{\ell+1}$. \square

The proof of Corollary 3.7(c), which is very similar to, though more tedious than, that of Corollary 3.7(b), is left to the reader.

4. PROOF OF THEOREM 3.3

4.1. The stabilizer of X_w . In this section we will apply Proposition 2.7 to obtain a description of the stabilizer $\text{Stab}(X_w) = \{g \in G \mid gX_w = X_w\}$ in terms of the data $(a(w), J(w))$. Review the definitions of the grading elements Z_1 and Z_w , and the (Z_1, Z_w) -bigraded decomposition $\mathfrak{g} = \oplus \mathfrak{g}_{j,k}$, in Sections 2.1 and 2.2.

Lemma 4.1. *The largest subalgebra $\mathfrak{g}_w \subset \mathfrak{g}$ containing \mathfrak{n}_w and such that $\mathfrak{g}_w \equiv \mathfrak{n}_w \pmod{\mathfrak{g}_{\geq 0}}$ is*

$$(4.2) \quad \mathfrak{g}_w = \mathfrak{n}_w \oplus \mathfrak{g}_{0,\geq 0} \oplus \mathfrak{g}_{1,\geq a}.$$

Proof. Let \mathfrak{g}' denote the right-hand side of (4.2). It is clear that \mathfrak{g}' is a subalgebra of \mathfrak{g} , and that $\mathfrak{g}' \equiv \mathfrak{n}_w \pmod{\mathfrak{g}_{\geq 0}}$. So $\mathfrak{g}' \subset \mathfrak{g}_w$. By construction $\mathfrak{g}_{0,\geq 0}$ is the stabilizer of \mathfrak{n}_w in \mathfrak{g}_0 , see Section 2.2. Consequently, $\mathfrak{g}_w \equiv \mathfrak{n}_w \oplus \mathfrak{g}_{0,\geq 0} \pmod{\mathfrak{g}_1}$. So it remains to see that,

$$(4.3) \quad \mathfrak{g}_{1,<a} \cap \mathfrak{g}_w = 0.$$

Assume the converse: suppose there exists a nonzero $\zeta \in \mathfrak{g}_{1,<a} \cap \mathfrak{g}_w$. Since \mathfrak{g}_1 is an irreducible \mathfrak{g}_0 -module, and $\mathfrak{g}_{0,\geq 0} \oplus \mathfrak{g}_{1,\geq a} \subset \mathfrak{g}_w$, we may assume without loss of generality that $\zeta \in \mathfrak{g}_{1,a-1}$. Then $[\zeta, \mathfrak{n}_w] \subset \mathfrak{g}_w$ holds if and only if $[\zeta, \mathfrak{g}_{-1,-a}] = 0$. In particular, $U = \{\zeta \in \mathfrak{g}_{1,a-1} \mid [\zeta, \mathfrak{g}_{-1,-a}] = 0\} \neq 0$.

The Jacobi identity implies U is a $\mathfrak{g}_{0,0}$ -module. So there exists $\Delta(U) \subset \Delta(\mathfrak{g}_{1,a-1})$ such that $U = \oplus_{\alpha \in \Delta(U)} \mathfrak{g}_\alpha$. Let $\gamma \in \Delta(U)$ be a highest $\mathfrak{g}_{0,0}$ -weight. Let $\tilde{\alpha}$ be the highest root of \mathfrak{g} . Then there exists a sequence $\sigma_1, \dots, \sigma_\ell \in \Sigma$ of simple roots such that $\gamma_i := \tilde{\gamma} - \sigma_1 - \cdots - \sigma_i$ is a root, for each $1 \leq i \leq \ell$, and $\gamma_\ell = \gamma$. Since γ is a highest $\mathfrak{g}_{0,0}$ -weight, and $\gamma_\ell + \sigma_\ell = \gamma_{\ell-1}$ is a root, σ_ℓ must lie in $\Delta(\mathfrak{g}_{0,1})$. In particular, $\gamma_{\ell-1} \in \Delta(\mathfrak{g}_{1,a})$. From $\gamma - \gamma_{\ell-1} = -\sigma_\ell$, we see that $0 \neq \mathfrak{g}_{-\sigma_\ell} = [\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma_{\ell-1}}] \subset [\mathfrak{g}_\gamma, \mathfrak{g}_{-1,-a}]$, implying $\mathfrak{g}_\gamma \not\subset U$, a contradiction. We conclude that (4.3) must hold. \square

Lemma 4.4. *The subalgebra \mathfrak{g}_w is parabolic.*

It is well-known that the stabilizer of the Schubert variety X_w is parabolic; see, for example, [1] or [3]. The algebra \mathfrak{g}_w is a nonstandard parabolic; that is, \mathfrak{g}_w does not contain the fixed Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. It can be shown that $\Sigma_w = \Sigma(\mathfrak{g}_{0,0}) \cup -\Pi(\mathfrak{g}_{1,a}) = (w\Sigma) \cap \Delta(\mathfrak{g}_{\leq 0})$ is a system of simple roots for the semisimple $[\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_0]$.

Proof. Set $\tilde{Z}_w = Z_w - aZ_1 \in \mathfrak{h}$, and $t = \max\{\alpha(\tilde{Z}_w) \mid \alpha \in \Delta\}$. The \tilde{Z}_w -graded decomposition of \mathfrak{g} is

$$(4.5) \quad \mathfrak{g} = \bigoplus_{s=-t}^t \tilde{\mathfrak{g}}_s \quad \text{where} \quad \tilde{\mathfrak{g}}_s := \{\zeta \in \mathfrak{g} \mid [\tilde{Z}_w, \zeta] = s\zeta\}.$$

Then

$$(4.6) \quad \mathfrak{g}_w = \tilde{\mathfrak{g}}_{\geq 0},$$

and is therefore parabolic by [4, Theorem 3.2.1(2)]. \square

The following is a corollary of Lemma 4.1.

Corollary 4.7. *The stabilizer of X_w in G is the parabolic subgroup G_w associated with \mathfrak{g}_w .*

Proposition 4.8 (Brion–Polo [3]). *The smooth locus X_w^0 of X_w is the orbit of $o \in X$ under the stabilizer G_w .*

4.2. Lemmas. We begin with a proof of Lemma 3.2. The remainder of the section is then devoted to the proof of Theorem 3.3; the theorem is an immediate corollary of Lemmas 4.10 and 4.12.

Remark 4.9. One important consequence of Remark 2.3 is that given a set $\Phi \subset \Delta(\mathfrak{g}_1)$, there exists $w \in W^p$ such that $\Delta(w) = \Phi$ if and only if $\Delta^+ \setminus \Phi$ is closed. For details see [14, Section 2.3].

Proof of Lemma 3.2. By Remark 2.3 it suffices to show that

$$\begin{aligned} \Phi(w, \varepsilon) &:= \Delta^+ \setminus \{\Delta(w) \setminus \Delta(w, \varepsilon)\} = \{\Delta^+ \setminus \Delta(w)\} \sqcup \Delta(w, \varepsilon) \\ &= \Delta(\mathfrak{g}_{1,>a}) \sqcup \Delta^+(\mathfrak{g}_{0,\geq 0}) \sqcup \Delta(w, \varepsilon) \end{aligned}$$

is closed. By Remark 2.3, the set $\Delta(\mathfrak{g}_{1,>a}) \sqcup \Delta(w, \varepsilon) \subset \Delta(\mathfrak{g}_1)$ is closed. Similarly, by Remark 4.9, the set $\Delta^+ \setminus \Delta(w) = \Delta(\mathfrak{g}_{1,>a}) \sqcup \Delta^+(\mathfrak{g}_{0,\geq 0})$ is closed. So it remains to show that given roots $\nu \in \Delta(w, \varepsilon)$ and $\beta \in \Delta^+(\mathfrak{g}_{0,\geq 0})$ such that $\nu + \beta$ is also a root, it is the case that $\nu + \beta \in \Phi(w, \varepsilon)$. There are two cases to consider: either $\nu = \varepsilon$, or $\nu = \varepsilon + \nu'$ for some $\nu' \in \Delta(\mathfrak{g}_{0,1})$.

(I) Assume $\nu = \varepsilon$ and $\nu + \beta \in \Delta$.

- If $\beta \in \Delta(\mathfrak{g}_{0,>1})$, then $\nu + \beta \in \Delta(\mathfrak{g}_{1,>a}) \subset \Phi(w, \varepsilon)$.
- If $\beta \in \Delta(\mathfrak{g}_{0,1})$, then $\nu + \beta \in \Delta(w, \varepsilon) \subset \Phi(w, \varepsilon)$.
- If $\beta \in \Delta^+(\mathfrak{g}_{0,0})$, then $\nu + \beta = \varepsilon + \beta$ cannot be a root because ε is a highest $\mathfrak{g}_{0,0}$ -weight.

(II) Assume $\nu = \varepsilon + \nu' \in \Delta(\mathfrak{g}_{1,a})$ and $\nu + \beta \in \Delta$.

- If $\beta \in \Delta(\mathfrak{g}_{0,>0})$, then $\nu + \beta \in \Delta(\mathfrak{g}_{1,>a}) \subset \Phi(w, \varepsilon)$.

- If $\beta \in \Delta^+(\mathfrak{g}_{0,0})$, then $\nu + \beta = \varepsilon + \nu' + \beta \in \Delta$; therefore,

$$\begin{aligned} \{0\} \neq \mathfrak{g}_{\nu+\beta} &= [\mathfrak{g}_\nu, \mathfrak{g}_\beta] = [[\mathfrak{g}_\varepsilon, \mathfrak{g}_{\nu'}], \mathfrak{g}_\beta] \\ &= [[\mathfrak{g}_\beta, \mathfrak{g}_{\nu'}], \mathfrak{g}_\varepsilon] + [[\mathfrak{g}_\varepsilon, \mathfrak{g}_\beta], \mathfrak{g}_{\nu'}]. \end{aligned}$$

Because ε is a highest $\mathfrak{g}_{0,0}$ -weight, the bracket $[\mathfrak{g}_\varepsilon, \mathfrak{g}_\beta]$ is zero. This forces $[\mathfrak{g}_\beta, \mathfrak{g}_{\nu'}]$ to be nonzero. Equivalently, $\beta + \nu' \in \Delta(\mathfrak{g}_{0,1})$. Thus $\nu + \beta = \varepsilon + (\nu' + \beta) \in \Delta(w, \varepsilon)$. \square

Lemma 4.10. *The $\{\Delta(w_\varepsilon) \mid \varepsilon \in \Pi_{1,a-1}\}$ is precisely the set of $\Delta(w_1) \subset \Delta(w)$, with $w_1 \in W^\mathfrak{p}$, that are maximal with the property that*

$$(4.11) \quad \Delta(w) \setminus \Delta(w_1) \not\subset \Delta(\mathfrak{g}_{1,a}).$$

Lemma 4.12. *Given $w_1 < w$, we have $Y_{w_1} \subset \text{Sing}(Y_w)$ if and only if $\Delta(w) \setminus \Delta(w_1) \not\subset \Delta(\mathfrak{g}_{1,a})$.*

Given a root γ , let $r_\gamma \in W$ denote the associated reflection. In order to prove Lemmas 4.10 and 4.12 we first recall

Lemma 4.13. *Let $w_1, w \in W^\mathfrak{p}$ be elements of the Hasse diagram of a cominuscule G/P . Then $w_1 \leq w$ if and only if $\Delta(w_1) \subset \Delta(w)$. In this case there is an ordering $\{\gamma_1, \gamma_2, \dots, \gamma_m\}$ of the elements of $\Delta(w) \setminus \Delta(w_1)$ so that $w_{\ell+1} = r_{\gamma_\ell} w_\ell \in W^\mathfrak{p}$, $w_{m+1} = w$ and $\Delta(w_{\ell+1}) = \Delta(w_1) \sqcup \{\gamma_1, \dots, \gamma_\ell\}$.*

Proof. This well-known result may be deduced from Propositions 3.2.12(5) and 3.2.15(3) of [4]. Note that their Φ_w is our $\Delta(w)$. \square

Corollary 4.14. *The Weyl group element $w_1 = r_\varepsilon w_\varepsilon$ is an element of $W^\mathfrak{p}$ and $\Delta(r_\varepsilon w_\varepsilon) = \Delta(w_\varepsilon) \sqcup \{\varepsilon\}$. Moreover, there exists an ordering $\{\nu_1, \dots, \nu_m\}$ of the elements of $\Delta(w, \varepsilon) \setminus \{\varepsilon\}$ so that $w_{\ell+1} := r_{\nu_\ell} \dots r_{\nu_1} w_1 \in W^\mathfrak{p}$ and $\Delta(w_{\ell+1}) = \Delta(w_\varepsilon) \sqcup \{\varepsilon, \nu_1, \dots, \nu_\ell\}$, for all $1 \leq \ell \leq m$.*

Proof. It suffices to observe that, in the ordering of the roots $\Delta(w, \varepsilon) = \Delta(w) \setminus \Delta(w_\varepsilon)$ given by Lemma 4.13, it is necessarily the case that $\gamma_1 = \varepsilon$. By Remark 4.9, the set $\Phi = \Delta^+ \setminus \Delta(r_{\gamma_1} w_\varepsilon)$ is closed. Suppose that $\gamma_1 \neq \varepsilon$. Then, by the definition (3.1) of $\Delta(w, \varepsilon)$, there exists $\mu \in \Delta(\mathfrak{g}_{0,1})$ such that $\gamma_1 = \varepsilon + \mu$. However, $\varepsilon, \mu \in \Phi$, while $\gamma_1 \notin \Phi$, contradicting the closure of Φ . \square

Proof of Lemma 4.10. Recall (Proposition 2.7) that $\Delta(w) = \Delta(\mathfrak{g}_{1, \leq a})$. By Lemma 3.2, $\Delta(w) \setminus \Delta(w_\varepsilon) = \Delta(w, \varepsilon)$. The definition (3.1) yields $\Delta(w, \varepsilon) \cap \Delta(\mathfrak{g}_{1, < a}) = \{\varepsilon\}$. So $\Delta(w, \varepsilon) \not\subset \Delta(\mathfrak{g}_{1,a})$. To see that $\Delta(w_\varepsilon)$ is maximal with respect to (4.11), recall (Remark 4.9) that $\Delta^+ \setminus \Delta(w_\varepsilon)$ is closed; this forces $\Delta(w, \varepsilon) \subset \Delta^+ \setminus \Delta(w_\varepsilon)$.

Conversely, suppose that $\Delta(w_1) \subset \Delta(w)$ satisfies (4.11). Fix $\mu = \mu_0 \in \Delta(\mathfrak{g}_{1, < a}) \setminus \Delta(w_1)$. There exists a sequence of simple roots $\sigma_1, \dots, \sigma_\ell \in \Sigma$ such that each $\mu_i := \mu + \sigma_1 + \dots + \sigma_i$ is a root, for all $1 \leq i \leq \ell$, and μ_ℓ is the highest root of \mathfrak{g} . Since both μ and μ_ℓ lie in $\Delta(\mathfrak{g}_1)$, the simple roots σ_i must lie in $\Delta^+(\mathfrak{g}_0)$. It follows from Remark 4.9 that each $\mu_i \in \Delta^+ \setminus \Delta(w_1)$. Moreover, since μ_ℓ is the highest root of \mathfrak{g} , at least one of the μ_i is an element of $\Delta(\mathfrak{g}_{1,a-1})$. Let $U \subset \mathfrak{g}_{1,a-1}$ be the irreducible $\mathfrak{g}_{0,0}$ -submodule containing \mathfrak{g}_{μ_i} . Let $\varepsilon \in \Delta(U)$ be the highest $\mathfrak{g}_{0,0}$ -weight of U . There exists a second sequence of simple roots $\sigma'_1, \dots, \sigma'_m \in \Sigma(\mathfrak{g}_{0,0})$ such that each $\mu_{i,k} := \mu_i + \sigma'_1 + \dots + \sigma'_k \in \Delta(U)$, with $1 \leq k \leq m$, and $\mu_{i,m} = \varepsilon$. Since $\mu_i \in \Delta^+ \setminus \Delta(w_1)$, Remark 4.9 implies $\varepsilon \in \Delta^+ \setminus \Delta(w_1)$.

As in the first paragraph of this proof, Remark 4.9 forces $\Delta(w, \varepsilon) \subset \Delta^+ \setminus \Delta(w_1)$. Thus, $\Delta(w_1) \subset \Delta(w_\varepsilon)$. \square

Proof of Lemma 4.12. First we will show that the lemma is equivalent to (4.15c). Recall from (2.5) that $X_w = wY_w$. So the lemma is equivalent to

$$(4.15a) \quad wY_{w_1} \subset \text{Sing}(X_w) \quad \text{if and only if} \quad \Delta(w) \setminus \Delta(w_1) \not\subset \Delta(\mathfrak{g}_{1,a}).$$

By Lemma 4.13 we have $w = \tau w_1$, where $\tau = r_{\gamma_m} \cdots r_{\gamma_2} r_{\gamma_1}$ and $\{\gamma_1, \gamma_2, \dots, \gamma_m\} = \Delta(w) \setminus \Delta(w_1)$. So $wY_{w_1} = \tau X_{w_1}$, and (4.15a) is equivalent to

$$(4.15b) \quad \tau X_{w_1} \subset \text{Sing}(X_w) \quad \text{if and only if} \quad \Delta(w) \setminus \Delta(w_1) \not\subset \Delta(\mathfrak{g}_{1,a}).$$

By Lemma 4.13 we have $\Delta(w_1) \subset \Delta(w)$. Equations (2.4) and (4.2) then imply $\mathfrak{n}_{w_1} \subset \mathfrak{n}_w \subset \mathfrak{g}_w$, and therefore $N_{w_1} \subset N_w \subset G_w$. By Proposition 4.8, $G_w \cdot o = X_w \setminus \text{Sing}(X_w) = X_w^0$. So $N_{w_1} \cdot o \subset G_w \cdot o = X_w^0$. Since $X_{w_1} = \overline{N_{w_1} \cdot o}$, we see that $\tau X_{w_1} \subset \text{Sing}(X_w)$ if and only if $\tau N_{w_1} \cdot o \not\subset G_w \cdot o$. Therefore, (4.15b) is equivalent to

$$(4.15c) \quad \tau N_{w_1} \cdot o \not\subset G_w \cdot o \quad \text{if and only if} \quad \Delta(w) \setminus \Delta(w_1) \not\subset \Delta(\mathfrak{g}_{1,a}).$$

Let $\gamma \in \Delta(w) \setminus \Delta(w_1)$. As an element of $W = N_G(H)/H$, the reflection r_γ is represented by $\exp(\xi)\exp(\zeta)\exp(\xi) \in N_G(H)$, where the $\xi \in \mathfrak{g}_\gamma$ and $\zeta \in \mathfrak{g}_{-\gamma}$ are scaled so that $\gamma([\xi, \zeta]) = -2$; see, for example, the proof of [4, Theorem 3.2.19(1)]. Lemma 4.1 and Corollary 4.7 imply that

$$(4.16) \quad r_\gamma \in G_w \quad \text{if and only if} \quad \gamma \in \Delta(\mathfrak{g}_{1,a}).$$

In this case, $r_\gamma N_{w_1} \subset r_\gamma G_w = G_w$. This establishes one direction of (4.15c): if $\Delta(w) \setminus \Delta(w_1) \subset \Delta(\mathfrak{g}_{1,a})$, then $\tau \in G_w$ and $\tau N_{w_1} \cdot o \subset G_w \cdot o = X_w \setminus \text{Sing}(X_w)$.

Suppose $\Delta(w) \setminus \Delta(w_1) \not\subset \Delta(\mathfrak{g}_{1,a})$. By Lemma 4.10 there exists $\varepsilon \in \Pi_{1,a-1}$ such that $\Delta(w_1) \subset \Delta(w_\varepsilon)$.

Claim. If $r_\varepsilon \notin G_w P$, then $\tau N_{w_1} \cdot o \not\subset G_w \cdot o$.

Assume that claim holds. Then to establish the second direction of (4.15c), it remains to show that the reflection $r_\varepsilon \in W$ can not be represented by an element $\tilde{p}p = G_w P$ with $\tilde{p} \in G_w$ and $p \in P$.

Recall the Z_1 -graded decomposition (2.1) of \mathfrak{g} . Let $G_0 := \{g \in G \mid \text{Ad}_g(\mathfrak{g}_j) \subset \mathfrak{g}_j\}$. Then G_0 is a closed subgroup of G with Lie algebra \mathfrak{g}_0 . By [4, Theorem 3.1.3], the map $\mathfrak{g}_1 \times G_0 \rightarrow P$ sending $(u, g) \mapsto \exp(u)g$ is a diffeomorphism. Likewise, recall the Z_w -graded decomposition (4.5) of \mathfrak{g} . Again, $\tilde{G}_0 := \{g \in G \mid \text{Ad}_g(\tilde{\mathfrak{g}}_k) \subset \tilde{\mathfrak{g}}_k\}$ is a closed subgroup of G with Lie algebra $\tilde{\mathfrak{g}}_0$. By (4.6), $\mathfrak{g}_w = \tilde{\mathfrak{g}}_{\geq 0} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_+$, and [4, Theorem 3.1.3] implies that the map $\tilde{\mathfrak{g}}_+ \times \tilde{G}_0 \rightarrow G_w$ sending $(\tilde{u}, \tilde{g}) \mapsto \exp(\tilde{u})\tilde{g}$ is a diffeomorphism.

We will argue by contradiction, supposing that $\tilde{p}p \in G_w P$ represents the reflection r_ε . In particular, $\text{Ad}_{\tilde{p}p} : \mathfrak{g} \rightarrow \mathfrak{g}$ preserves the Cartan subalgebra \mathfrak{h} . Write $\tilde{p} = \exp(\tilde{u})\tilde{g}$, with $\tilde{u} \in \tilde{\mathfrak{g}}_+$ and $\tilde{g} \in \tilde{G}_0$, and $p = \exp(u)g$, with $g \in G_0$ and $u \in \mathfrak{g}_1$. Fix $H \in \mathfrak{h}$, and define $h_s \in \tilde{\mathfrak{g}}_s$ by $\text{Ad}_p H = \sum_{s=-t}^t \tilde{h}_s$, and set $\tilde{h}_s = \text{Ad}_{\tilde{g}} h_s \in \tilde{\mathfrak{g}}_s$. Define $\tilde{h}_{s,r} \in \tilde{\mathfrak{g}}_r$ by $\text{Ad}_{\exp(\tilde{u})} \tilde{h}_s = \tilde{h}_{s,s} + \tilde{h}_{s,s+1} + \cdots + \tilde{h}_{s,t}$, and note that $\tilde{h}_{s,s} = \tilde{h}_s$. Then

$$\text{Ad}_{\tilde{p}p} H = \sum_{s=-t}^t \sum_{r=s}^t \tilde{h}_{s,r} = \sum_{r=-t}^t \tilde{H}_r,$$

where $\tilde{H}_r := \sum_{s=-t}^r \tilde{h}_{s,r} \in \tilde{\mathfrak{g}}_r$.

Since $\text{Ad}_{\tilde{p}p}$ preserves \mathfrak{h} , and $\mathfrak{h} \subset \tilde{\mathfrak{g}}_0$, it must be the case that

$$(4.17) \quad \tilde{H}_0 \in \mathfrak{h}, \quad \text{and} \quad \tilde{H}_r = 0, \quad \text{when } r \neq 0.$$

In particular, $\tilde{H}_{-t} = \tilde{h}_{-t,-t} = \tilde{h}_{-t} = 0$. This in turn yields $\tilde{h}_{-t,r} = 0$ for all r . Moreover, since $\tilde{h}_{-t} = \text{Ad}_{\tilde{g}} h_{-t}$, we also have $h_{-t} = 0$. Next, $0 = \tilde{H}_{1-t} = \tilde{h}_{-t,1-t} + \tilde{h}_{1-t,1-t} = \tilde{h}_{1-t}$. As above, this implies $\tilde{h}_{1-t,r} = 0$, for all r , and $h_{1-t} = 0$. Continuing by induction, we see that

$$(4.18) \quad h_s = 0 \quad \text{for all } s < 0.$$

In particular, $\text{Ad}_p H \in \tilde{\mathfrak{g}}_{\geq 0}$. Our choice of $H \in \mathfrak{h}$ was arbitrary, so $\text{Ad}_p \mathfrak{h} \subset \tilde{\mathfrak{g}}_{\geq 0} = \mathfrak{g}_w$. This implies $p \in G_w$. In particular, $\tilde{p}p \in G_w$.

This yields a contradiction as follows. Note that $\mathfrak{g}_{-\varepsilon} \subset \tilde{\mathfrak{g}}_1$. So given any $q \in G_w$, we have $\text{Ad}_q \mathfrak{g}_{-\varepsilon} \subset \tilde{\mathfrak{g}}_{\geq 1}$. On the other hand, $\mathfrak{g}_{\varepsilon} \subset \tilde{\mathfrak{g}}_{-1}$, and $\text{Ad}_{r_{\varepsilon}}(\mathfrak{g}_{-\varepsilon}) = \mathfrak{g}_{\varepsilon}$. Therefore, there exists no element $q \in G_w$ such that $\text{Ad}_q(\mathfrak{g}_{-\varepsilon}) = \tilde{\mathfrak{g}}_{\varepsilon}$. Modulo the claim, this completes the proof of Lemma 4.12.

Proof of claim. By Lemma 4.13, $w_1 \leq w_{\varepsilon}$. Therefore, $Y_{w_1} \subset Y_{w_{\varepsilon}}$. So to see that $Y_{w_1} \subset \text{Sing}(Y_w)$, it suffices to show that $Y_{w_{\varepsilon}} \subset \text{Sing}(Y_w)$. Equivalently, as discussed above, $\tau N_{w_{\varepsilon}} \cdot o \notin G_w \cdot o$, where $\tau = r_{\nu_m} \cdots r_{\nu_1} r_{\varepsilon}$ is as given by Corollary 4.14. Since $\nu_{\ell} \in \Delta(\mathfrak{g}_{1,\mathbf{a}})$, we have $r_{\nu_{\ell}} \in G_w$, by (4.16). So $\tau N_{w_{\varepsilon}} \cdot o \notin G_w \cdot o$ if and only if $r_{\varepsilon} N_{w_{\varepsilon}} \cdot o \notin G_w \cdot o$. In particular, to see that $\tau N_{w_{\varepsilon}} \cdot o \notin G_w \cdot o$, it suffices to show that $r_{\varepsilon} \cdot o \notin G_w \cdot o$. Lifting to G , the latter is equivalent to $r_{\varepsilon} \notin G_w P$. \square

APPENDIX A. GEOMETRIC DESCRIPTIONS OF $X_{\mathbf{a},\mathbf{J}}$

The appendix provides a dictionary to translate between the \mathbf{a}, \mathbf{J} characterization of a Schubert variety $X_w \subset X$, and the more familiar geometric description. The dictionary was first given in [14] (in the case that $G = A_n$) and [13] (for all simple G). The appendix does not treat the quadric hypersurfaces; see [13] for the dictionary in that case.

Notation. Given a basis $\{e_1, \dots, e_n\}$ of a vector space V , let $\{e^1, \dots, e^n\}$ denote the dual basis of V^* . Set $e_b^a = e_b \otimes e^a \in \text{End}(V)$ for all $1 \leq a, b \leq n$.

A.1. Grassmannians $\text{Gr}(\mathbf{i}, n+1) = A_n/P_{\mathbf{i}}$. There is a bijection between $W^{\mathbf{p}} \setminus \{1, w_0\}$ and pairs \mathbf{a}, \mathbf{J} such that $\mathbf{J} = \{\mathbf{j}_{\mathbf{p}}, \dots, \mathbf{j}_1, \mathbf{k}_1, \dots, \mathbf{k}_{\mathbf{q}}\} \subset \{1, \dots, n\} \setminus \{\mathbf{i}\}$ is ordered so that

$$1 \leq \mathbf{j}_{\mathbf{p}} < \dots < \mathbf{j}_1 < \mathbf{i} < \mathbf{k}_1 < \dots < \mathbf{k}_{\mathbf{q}} \leq n,$$

and satisfying $\mathbf{p}, \mathbf{q} \in \{\mathbf{a}, \mathbf{a}+1\}$; see [14, Corollary 3.17]. (Beware, these \mathbf{p}, \mathbf{q} do not agree with those of [14], cf. Remark 2.9.) For convenience we set

$$\mathbf{j}_{\mathbf{p}+1} := 0, \quad \mathbf{j}_0 := \mathbf{i} =: \mathbf{k}_0, \quad \mathbf{k}_{\mathbf{q}+1} := n+1.$$

Fix a basis $\{e_1, \dots, e_{n+1}\}$ of \mathbb{C}^{n+1} . The abelian subalgebra \mathfrak{g}_{-1} is spanned by the root vectors $\{e_{\ell}^k \mid 1 \leq k \leq \mathbf{i} < \ell \leq n+1\}$; the corresponding roots are $-(\alpha_k + \dots + \alpha_{\ell-1})$. Define a filtration $F_{\mathbf{a}+1} \subset F_{\mathbf{a}} \subset \dots \subset F_1 \subset F_0$ of \mathbb{C}^{n+1} by

$$F_{\ell} = \langle e_1, e_2, \dots, e_{\mathbf{j}_{\ell}}, e_{\mathbf{i}+1}, e_{\mathbf{i}+2}, \dots, e_{\mathbf{k}_m} \rangle \quad \text{with } \ell + m = \mathbf{a} + 1.$$

Set $o = [e_1 \wedge \dots \wedge e_{\mathbf{i}}] \in \mathbb{P}(\wedge^{\mathbf{i}} \mathbb{C}^{n+1})$. Then

$$X_w = \{E \in \text{Gr}(\mathbf{i}, n+1) \mid \dim(E \cap F_{\ell}) \geq \mathbf{j}_{\ell}, \ 0 \leq \ell \leq \mathbf{a}+1\}.$$

Example. Consider $X = \text{Gr}(5, 13) \simeq A_{12}/P_5$. The marking $\mathbf{J} = \{2, 3, 7, 9, 12\}$ and integer $\mathbf{a} = 2$ define the filtration $F_3 \subset F_2 \subset F_1 \subset F_0$ as

$$\begin{aligned} F_3 &= \langle 0 \rangle, & F_2 &= \langle e_1, e_2, e_6, e_7 \rangle, \\ F_1 &= \langle e_1, \dots, e_3, e_6, \dots, e_9 \rangle, & F_0 &= \langle e_1, \dots, e_5, e_6, \dots, e_{12} \rangle. \end{aligned}$$

The associated Schubert variety is the set of all $E \in \text{Gr}(5, 13)$ such that

$$\dim(E \cap F_3) \geq 0, \quad \dim(E \cap F_2) \geq 2, \quad \dim(E \cap F_1) \geq 3, \quad \dim(E \cap F_0) \geq 5.$$

Partitions versus \mathbf{a}, \mathbf{J} . It is well-known that Schubert varieties in $\text{Gr}(\mathbf{i}, n+1)$ are indexed by partitions

$$\pi = (a_1, \dots, a_{\mathbf{i}}) \in \mathbb{Z}^{\mathbf{i}} \quad \text{such that} \quad n+1-\mathbf{i} \geq a_1 \geq a_2 \geq \dots \geq a_{\mathbf{i}} \geq 0.$$

Fix a flag $0 \subset F^1 \subset F^2 \subset \dots \subset F^{n+1}$. The corresponding Schubert variety

$$X_\pi := \{E \in \text{Gr}(\mathbf{i}, n+1) \mid \dim(E \cap F^{n+1-\mathbf{i}+j-a_j}) \geq j\}$$

is of codimension $|\pi| := a_1 + \dots + a_{\mathbf{i}}$. The relationship between the partition and (\mathbf{a}, \mathbf{J}) descriptions of a Schubert variety is given by the proposition below. It will be convenient to write the partition as $\pi = (c_1^{b_1}, \dots, c_r^{b_r})$, where $c_1 > c_2 > \dots > c_r > 0$, and $b_\ell \geq 1$ for all ℓ . Note that $c_1 \leq n+1-\mathbf{i}$ and $b_1 + \dots + b_r \leq \mathbf{i}$. Let π^* denote the conjugate partition. The following is [14, Proposition 3.30].

Proposition A.1 ([14]). *Given $\pi = (c_1^{b_1}, \dots, c_r^{b_r})$, the Schubert variety X_π has (\mathbf{a}, \mathbf{J}) -description with $\mathbf{a}(w) = r^* - 1$ and*

$$\mathbf{J}(\pi) = \{b_1, b_1 + b_2, \dots, \sum b_s, n+1-c_1, n+1-c_2, \dots, n+1-c_r\} \setminus \{\mathbf{i}\}.$$

In particular,

- $\mathbf{p} = \mathbf{a}$ if $c_1 < n+1-\mathbf{i}$, and $\mathbf{p} = \mathbf{a} + 1$ if $c_1 = n+1-\mathbf{i}$;
- $\mathbf{q} = \mathbf{a}$ if $\sum b_j < \mathbf{i}$, and $\mathbf{q} = \mathbf{a} + 1$ if $\sum b_j = \mathbf{i}$.

Remark. From the proposition and [1, Theorem 9.3.1] we deduce that the integer $\mathbf{a}(w)$ is the number of irreducible components of $\text{Sing}(X_w)$. See also Corollary 3.6.

A.2. Lagrangian Grassmannians $\text{LG}(n, 2n) = C_n/P_n$. There exists a bijection between $W^{\mathbf{p}} \setminus \{1, w_0\}$ and pairs $\mathbf{a} \geq 0$ and $\mathbf{J} = \{j_{\mathbf{p}}, \dots, j_1\} \subset \{1, \dots, n-1\}$ satisfying

$$1 \leq j_{\mathbf{p}} < \dots < j_1 \leq n-1$$

and $|\mathbf{J}| = \mathbf{p} \in \{\mathbf{a}, \mathbf{a} + 1\}$; see [14, Corollary 3.17]. (Note that these j_ℓ have the opposite order of those in [14].) For convenience we set

$$j_{\mathbf{p}+1} := 0, \quad j_0 := n.$$

To describe the Schubert variety X_w , fix a nondegenerate skew-symmetric bilinear form (\cdot, \cdot) on \mathbb{C}^{2n} , and basis $\{e_1, \dots, e_{2n}\}$ of \mathbb{C}^{2n} satisfying $(e_a, e_b) = 0 = (e_{n+a}, e_{n+b})$ and $1 = (e_a, e_{n+b}) = \delta_{ab}$ for all $1 \leq a, b \leq n$. The abelian subalgebra \mathfrak{g}_{-1} (which may be identified with n -by- n symmetric matrices) is spanned by the root vectors $\{e_{n+k}^j + e_{n+j}^k \mid 1 \leq j \leq k \leq n\}$, with roots $-(\alpha_j + \dots + \alpha_{k-1}) - 2(\alpha_k + \dots + \alpha_n)$. Define a filtration $F_{\mathbf{p}} \subset \dots \subset F_1 \subset F_0$ of \mathbb{C}^{2n} by

$$F_\ell = \langle e_1, \dots, e_{j_\ell}, e_{n+j_{\ell}+1}, \dots, e_{2n} \rangle \quad \text{with} \quad \ell + m = \mathbf{a} + 1.$$

Set $o = [e_1 \wedge \dots \wedge e_n] \in \mathbb{P}(\wedge^n \mathbb{C}^{2n})$. Then

$$X_w = \{E \in \text{LG}(n, 2n) \mid \dim(E \cap F_\ell) \geq j_\ell, \forall 0 \leq \ell \leq \mathbf{p}\}.$$

Example. Consider $X = \text{LG}(5, 10) \simeq C_5/P_5$. The marking $J = \{1, 2, 4\}$ and integer $\mathbf{a} = 3$ define the filtration $F_3 \subset F_2 \subset F_1 \subset F_0$ of \mathbb{C}^{10} as

$$\begin{aligned} F_3 &= \langle e_1, e_{10} \rangle, & F_2 &= \langle e_1, e_2, e_8, e_9, e_{10} \rangle, \\ F_1 &= \langle e_1, \dots, e_4, e_7, \dots, e_{10} \rangle, & F_0 &= \mathbb{C}^{10}. \end{aligned}$$

The associated Schubert variety is the set of all $E \in \text{LG}(5, 10)$ such that

$$\dim(E \cap F_3) \geq 1, \quad \dim(E \cap F_2) \geq 2, \quad \dim(E \cap F_1) \geq 4.$$

A.3. Spinor varieties $\mathcal{S}_n = D_n/P_n$. Given $\mathbf{a} = \mathbf{a}(w)$ and $J = J(w)$, note that

$$\alpha_{n-1}(Z_w) = 0 \quad \text{if } n-1 \notin J, \quad \text{and} \quad \alpha_{n-1}(Z_w) = 1 \quad \text{if } n-1 \in J.$$

Define

$$r = \left\lceil \frac{1}{2}(\mathbf{a} + \alpha_{n-1}(Z_w)) \right\rceil = \begin{cases} \lceil \mathbf{a}/2 \rceil & \text{if } n-1 \notin J, \\ \lceil \mathbf{a}/2 \rceil + 1 & \text{if } n-1 \in J. \end{cases}$$

There exists a bijection between $W^{\mathfrak{p}} \setminus \{1, w_0\}$, and pairs $\mathbf{a} \geq 0$ and $J = \{j_{\mathfrak{p}}, \dots, j_1\} \subset \{1, \dots, n-1\}$, ordered so that

$$1 \leq j_{\mathfrak{p}} < \dots < j_1 \leq n-1,$$

and satisfying

$$|J| - \alpha_{n-1}(Z_w) \in \{\mathbf{a}, \mathbf{a} + 1\}, \quad \text{and} \quad 2 \leq j_r - j_{r+1} \quad \text{when } r > \alpha_{n-1}(Z_w);$$

see [14, Corollary 3.17]. (Note that these j_{ℓ} have the opposite order of those in [14].) For convenience we set

$$j_{\mathfrak{p}+1} := 0, \quad j_0 := n.$$

To describe the Schubert variety X_w , fix a nondegenerate symmetric bilinear form (\cdot, \cdot) on \mathbb{C}^{2n} and basis $\{e_1, \dots, e_{2n}\}$ of \mathbb{C}^{2n} satisfying $(e_a, e_b) = 0 = (e_{n+a}, e_{n+b})$ and $1 = (e_a, e_{n+b}) = \delta_{ab}$ for all $1 \leq a, b \leq n$. The abelian subalgebra \mathfrak{g}_{-1} (which may be identified with n -by- n skew-symmetric matrices) is spanned by root vectors $\{e_{n+k}^j - e_{n+j}^k \mid 1 \leq j < k \leq n\}$. The corresponding roots are $-(\alpha_j + \dots + \alpha_{n-2}) - \alpha_n$, if $k = n$; $-(\alpha_j + \dots + \alpha_n)$, if $k = n-1$; and $-(\alpha_j + \dots + \alpha_{k-1}) - 2(\alpha_k + \dots + \alpha_{n-2}) - \alpha_{n-1} - \alpha_n$, if $k < n-1$. Define a filtration $F_{\mathfrak{p}} \subset \dots \subset F_1 \subset F_0$ of \mathbb{C}^{2n} by

$$F_{\ell} = \langle e_1, \dots, e_{j_{\ell}}, e_{n+j_{\ell}+1}, \dots, e_{2n} \rangle$$

with

$$\ell + m = \begin{cases} \mathbf{a} + 1 & \text{if } n-1 \notin J, \\ \mathbf{a} + 2 & \text{if } n-1 \in J \end{cases} = \mathbf{a} + 1 + \alpha_{n-1}(Z_w).$$

Set $o = [e_1 \wedge \dots \wedge e_n] \in \mathbb{P}(\wedge^n \mathbb{C}^{2n})$. Then

$$X_w = \{E \in \mathcal{S}_n \mid \dim(E \cap F_{\ell}) \geq j_{\ell}, \forall 0 \leq \ell \leq \mathfrak{p}\}.$$

A.4. The exceptional cases. Figures 1 and 2 (pages 18 and 19) are respectively the Hasse diagrams W^p of the Cayley plane E_6/P_6 and Freudenthal variety E_7/P_7 . Each node represents a Schubert class ξ_w and is labeled with the corresponding $a(w) : J(w)$ values, which we obtained with the assistance of [9]. The height of the node indicates the dimension of X_w ; in particular, the lowest node $o \in X$ is at height zero. Two nodes are connected if the Schubert variety associated with the lower node is a divisor of the Schubert variety associated with the higher node.

REFERENCES

- [1] Sara Billey and V. Lakshmibai. *Singular loci of Schubert varieties*, volume 182 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2000.
- [2] Sara C. Billey and Gregory S. Warrington. Maximal singular loci of Schubert varieties in $SL(n)/B$. *Trans. Amer. Math. Soc.*, 355(10):3915–3945 (electronic), 2003.
- [3] Michel Brion and Patrick Polo. Generic singularities of certain Schubert varieties. *Math. Z.*, 231(2):301–324, 1999.
- [4] Andreas Čap and Jan Slovák. *Parabolic geometries. I*, volume 154 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2009. Background and general theory.
- [5] Aurélie Cortez. Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire. *Adv. Math.*, 178(2):396–445, 2003.
- [6] Christian Kassel, Alain Lascoux, and Christophe Reutenauer. The singular locus of a Schubert variety. *J. Algebra*, 269(1):74–108, 2003.
- [7] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.
- [8] Shrawan Kumar. The nil Hecke ring and singularity of Schubert varieties. *Invent. Math.*, 123(3):471–506, 1996.
- [9] LiE. Computer algebra package for semisimple Lie algebra computations, www-math.univ-poitiers.fr/~maavl/LiE/.
- [10] L. Manivel. Generic singularities of schubert varieties, 2001. arXiv:math/0105239.
- [11] L. Manivel. Le lieu singulier des variétés de Schubert. *Internat. Math. Res. Notices*, (16):849–871, 2001.
- [12] Nicolas Perrin. The Gorenstein locus of minuscule Schubert varieties. *Adv. Math.*, 220(2):505–522, 2009.
- [13] C. Robles. Schur flexibility of cominuscule Schubert varieties. Preprint, 2012.
- [14] C. Robles and D. The. Rigid schubert varieties in compact hermitian symmetric spaces. *Selecta Mathematica, New Series*, pages 1–61, 2012. 10.1007/s00029-011-0082-y.
- [15] Hugh Thomas and Alexander Yong. A combinatorial rule for (co)minuscule Schubert calculus. *Adv. Math.*, 222(2):596–620, 2009.
- [16] A. V. Zelevinskii. Small resolutions of singularities of Schubert varieties. *Functional Anal. Appl.*, 17(2):142–144, 1983.

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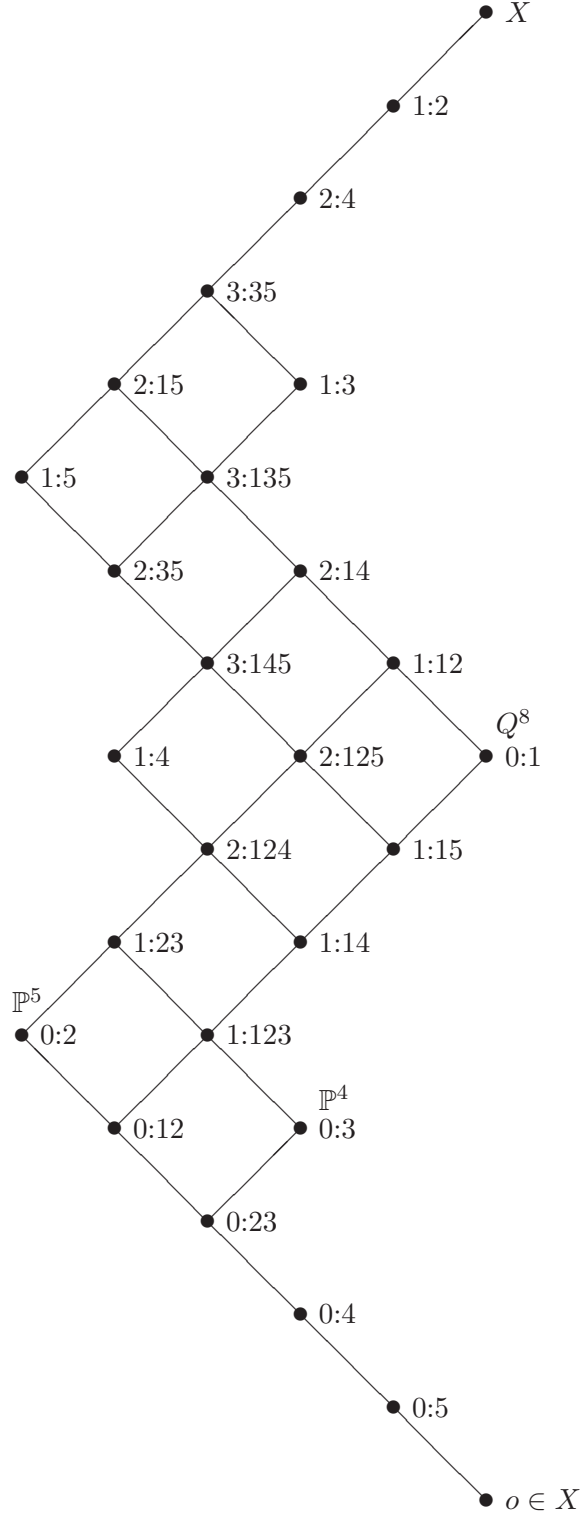
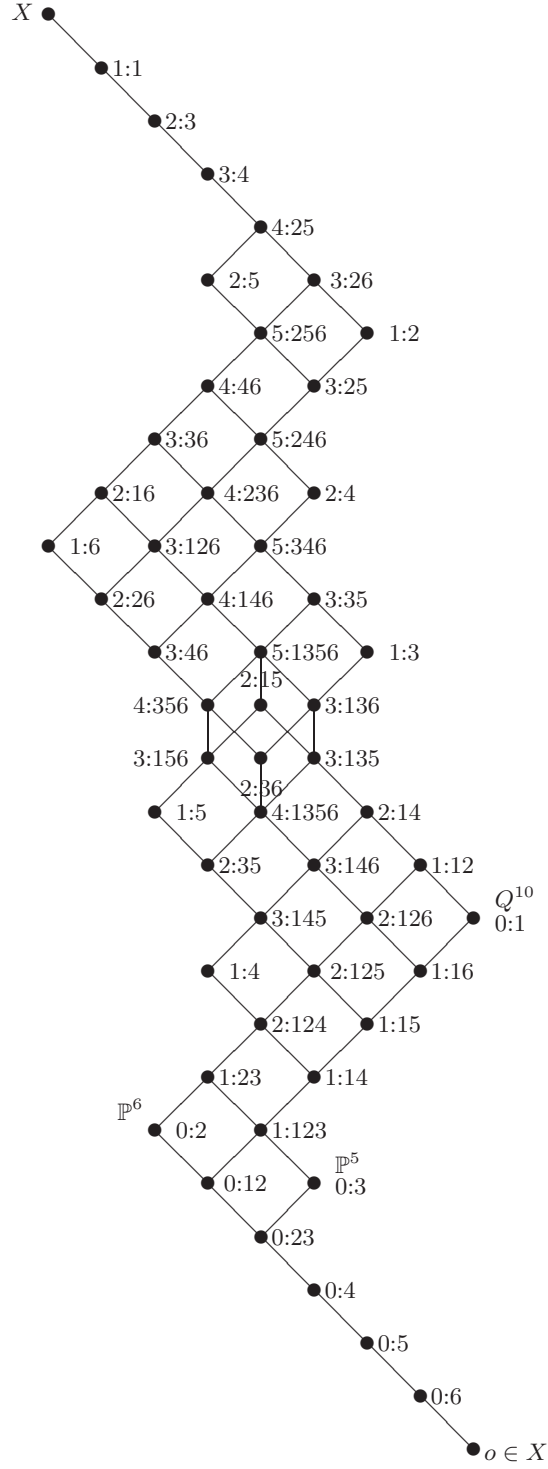


FIGURE 1. Hasse diagram of E_6/P_6 , each node labeled with the $a:J$ values.

FIGURE 2. Hasse diagram of E_7/P_7 , each node labeled with the $a:J$ values.